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date: June 21, 2000
Original March 24, 2000; Revised June 21, 2000 RRR

to: Distribution

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subject: Tutorial for Running Viscoelastic Flow Problems in GOMA (GT-014.1)

keywords: viscoelastic flow, multimode viscoelasticity, rheology, EVSS, DEVSS, Weissenberg number, viscoelastic theory, four-to-one contraction, inflow boundary conditions, automated continuation, discontinuous Galerkin

input records: Number of viscoplastic modes, Liquid Constitutive Equation, Polymer Constitutive Equation, Polymer Weighting, Adaptive Viscosity Scaling, Polymer Time Constant, Continuation, Hzero, Hfirst, Material property tag, Discontinuous Jacobian Formulation, PQ1, PQ2, Polymer Constitutive Equation=GIESEKUS, Coordinate System=CYLINDRICAL, EQ=stress11

Introduction

Plastics and polymers are ubiquitous in modern life from furniture and fabrics to automobiles and aircraft. Understanding the manufacturing of these products, in apparatus such as calendars, screw extruders and blow molders, can be very difficult. This is because the behavior of polymer solutions and melt is much more complex than ordinary fluids. Polymeric fluids can exhibit both fluid-like and solid-like characteristics, including viscous flow behavior, shear thinning or shear thickening, as a elasticity or memory. Their behavior is often counter-intuitive and highly non-linear. For this reason, numerical models are often used to explore the behavior of flowing polymers.

In GOMA, we have implemented several generalized Newtonian models (e.g. Carreau and Bingham plastic) that can simulate dilute polymer solutions and several viscoelastic models (e.g.Giesekus, Phan-Thien Tanner, and Maxwell) that are appropriate for concentrated solutions and melts. For the viscoelastic models, we can use up to eight different relaxation times allowing us to fit realistic data. The simulations are computationally intensive since an additional tensor is associated with equation each relaxation mode.

One problem that arises when trying to model the flow of viscoelastic fluids is known as the “High Weissenberg Number Problem.” The Weissenberg number is a dimensionless measure of the fluid elasticity. The “High Weissenberg Number Problem” is characterized by the loss of convergence of the numerical method as the amount of elasticity in the fluid is increased. Several of the best numerical methods available in the literature to circumvent the high Weissenberg number problem have been implemented in GOMA. These include the Elastic Viscous Stress Splitting (EVSS) method [Rajagopalan, et al., 1990], the extension of this method by Guenette and Fortin [1995] that greatly simplifies the stress equation and is sometimes termed the Discrete Elastic Viscous Stress Splitting (DEVSS) method, and the DEVSS method with the inclusion of an adaptive viscosity that stabilizes the momentum equation [Sun, et al., 1995]. This method is called the discrete adaptive viscoelastic stress splitting (DAVSS). In addition, we have also implemented the discontinuous Galerkin method to better handle the hyperbolic character of the stress equation [Fortin and Fortin, 1989 and Baaijens, 1994]. Following Sun, et al. [1995] we can combine the DAVSS method with a discontinuous Galerkin treatment for the stress equation, which can sometimes improve the convergence of the numerical scheme. Note that the original EVSS method is almost never used in GOMA since it creates a new stress variable that is hard to interpret.

The key to running viscoelastic flow problems is continuation in the Weissenberg number (We). The first step we take is to obtain a Newtonian solution, e.g. We = 0, on the mesh of interest with the stresses expressed as a tensor equation. (This is the so called “mixed method”, since we have a formulation that includes velocity, pressure and stress instead of the standard Newtonian formulation of velocity and pressure only.) We then use this as an initial guess to get a solution with a small non-zero value of We. In turn, we take this solution as an initial guess for a slightly higher value of We and then repeat this process until we reach the We of interest or until the numerical method fails to converge (usually the latter).

This tutorial will attempt to explain how to run a viscoelastic flow problem in GOMA. The demonstration problem we have chosen is the four to one contraction problem, results of which are widely available in the literature [see for instance Baaijens, 1998 or Purnode and Crochet, 1996]. This has become a classic benchmark problem since in a relatively simple geometry, we have a relatively complex flow. Representative features include a reentrant corner where the solution is singular [Renardy, 2000], a vortex in the corner before the contraction and a lip vortex downstream from the contraction.

Before attempting viscoelastic flow problems, the user should be proficient with running Newtonian problems in GOMA. Tutorial GT-001.4 should be consulted first. The name of the director containing this tutorial problem is ve_cont.

Theory

To understand the complexity of viscoelasticity, it is important to understand the equations. Here we write the viscoelastic equations in dimensionless variables for a Oldroyd-B fluid with a single relaxation time. First, we have a momentum equation that contains the viscoelastic stress and is parameterized by the Reynolds number (*Re*).

$$Re\frac{Dv}{Dt} + \nabla p + \nabla \bullet \tau = f \qquad\qquad Re = \frac{\rho \langle v \rangle d}{\eta_p}$$

Here ρ is the fluid density, <v> is the average velocity, d is a characteristic length scale and η_p is the polymer viscosity. Since we are usually dealing with highly viscous polymeric materials, the viscosity is always quite large making *Re* a nominal value that can be assumed to be zero.

We also have an incompressible continuity equation

$$\nabla \bullet v = 0$$

and a total stress (τ) that has a viscoelastic part (S) plus the Newtonian stress contribution, which is expressed in terms of the Stress number (*St*). *St* is the ratio of the solvent viscosity (η_s) to the polymer viscosity. Note, if *St* is zero we recover a Maxwell model from the Oldroyd-B model.

$$\tau = -St\dot{\gamma} + S \qquad St = \frac{\eta_s}{\eta_p}$$

The constitutive equation is dependent on the Weissenberg number (*We*), which is the dimensionless polymer time constant:

$$S + WeS_{(1)} = -\dot{\gamma} \qquad We = \frac{\lambda \langle u \rangle}{d}$$

For more details of the various constitutive equations for viscoelastic flow, please refer to Bird, et al. [1987].

Four to One Contraction Problem

Figure 1 is a schematic of the four to one contraction problem. The boundary conditions are quite standard: no slip on the solid boundary, symmetry about the center line, specified parabolic velocity profile on the inflow, and for the viscoelastic problem we also must specify the velocity profile on the outflow. The problem would be indeterminate on the outflow if only the natural boundary condition, viz. zero normal stress, were specified since it contains the pressure, velocity gradients and extra stress. If the stress at the inflow is known analytically or can be determined numerically, it is recommended that you specify the normal components at the inflow: the stress equation is hyperbolic and specifying at the beginning of a streamline such as the inflow may improve convergence [Renardy, 1985]. However, it is not always necessary to specify the stresses at inflow. A pressure datum is specified to anchor the pressure field, which is indeterminate to a constant because the equations only contain gradients of pressure.

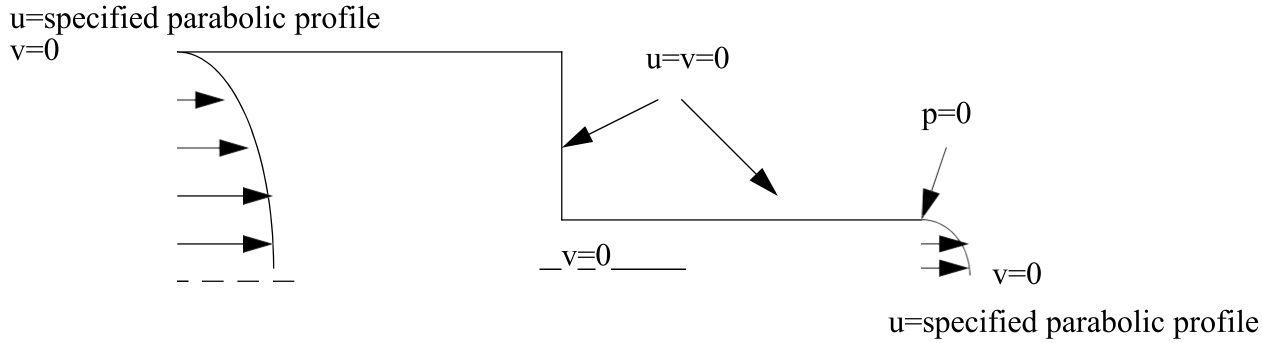


Figure 1. Boundary conditions for the Viscoelastic Four to One Contraction Problem

For an Oldroyd-B fluid, the exact solution for Poiseuille flow can be easily determined and used as the specified velocity profile. For other constitutive, such as Giesekus or Phan-Thien-Tanner models, no analytical solution is possible, so an ordinary differential equation (ODE) must be solved numerically to provide the inflow and outflow velocity profile. Such an ODE solver has been developed by Baer [1998] and is available for CRMPC members. Once determined, the numerical inflow or outflow profiles would then be input into GOMA using a TABLE_BC, examples of which can be found in the GOMA Manual [Schunk, et al., 1998].

The mesh for the problem is given in Figure 2. The FASTQ file used to create this mesh can be found in Appendix 1. The inflow is at x=-12 and the outflow is at x=16. A longer outflow section may be necessary for highly elastic fluids since the development length increases with the Weissenberg number.

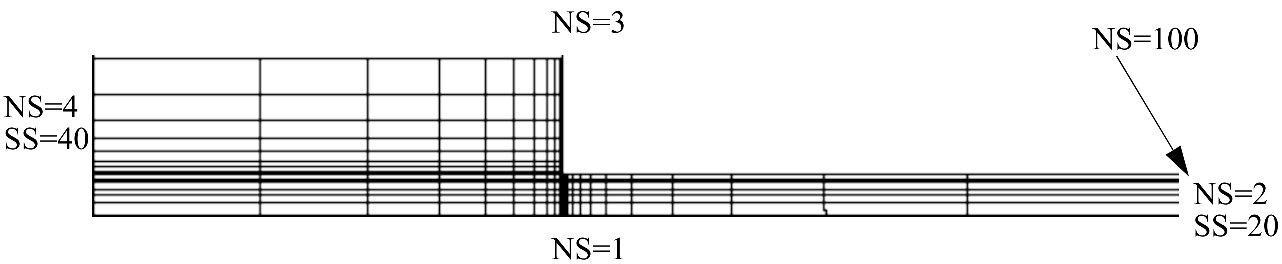


Figure 2. Mesh for Four to One Contraction Problem with Node Sets (NS) and Side Sets (SS) Indicated

2D Continuous Stress Maxwell Model Example Problem

The first test case we will look at is one of the simplest: the flow of a Maxwell fluid with a single relaxation time using a continuous stress interpolation in a planar 2D contraction.

Boundary Conditions

For the Maxwell fluid, the solution for Poiseuille flow can be computed analytically. If we work nondimensionally, assuming the average velocity is unity, the large channel has a height of 4, and the small channel a height of 1, then the inflow velocity and stress profiles are:

$$u = 1.5\left(1 - \frac{y^2}{16}\right)$$

$$v = 0$$

$$\tau_{xx} = 2We\left(\frac{du}{dy}\right)^2 = \frac{18}{256}We\,y^2$$

$$\tau_{xy} = \left(\frac{du}{dy}\right) = -\frac{3}{16}y$$

$$\tau_{yy} = 0$$

On the outflow, the boundary conditions are:

$$u = 6.0(1 - y^2)$$

$$v = 0$$

If we have a Weissenberg number, *We*, of 0.02 we get the following boundary condition section for our GOMA input file:

```
Number of BC = -1
# Centerline symmetry condition
BC = V     NS     1       0.
# No slip on the solid boundary
BC = U     NS     3       0.
BC = V     NS     3       0.
# Inflow velocities and stresses
BC = GD_LINEAR SS 40 R_MOMENTUM1  0 VELOCITY1      0  0.   1.
BC = GD_PARAB  SS 40 R_MOMENTUM1  0 MESH_POSITION2 0 -1.5  0. 0.09375
BC = V     NS     4       0.
BC = S22     NS     4     0.
BC = GD_LINEAR SS 40 R_STRESS11  0 POLYMER_STRESS11 0  0. 1.
BC = GD_PARAB  SS 40 R_STRESS11  0 MESH_POSITION2 0 0. 0. -0.00140625
# Outflow velocities
BC = GD_LINEAR SS 20 R_MOMENTUM1  0 VELOCITY1      0  0.   1.
BC = GD_PARAB  SS 20 R_MOMENTUM1  0 MESH_POSITION2 0 -6.   0. 6.
BC = V     NS     2       0.
# Pressure datum
BC = P     NS    100    0.
END OF BC
```

Note that GD conditions tend to be the simplest way of specifying the velocities and stresses that depend upon position. It would also be convenient to use APREPRO to calculate the inflow stress field.

Equation Specifications

For a planar contraction flow using the DEVSS method, we need two momentum equations, a continuity equation, three stress equations and four velocity gradient equations [see Guenette and Fortin, 1995 for details of the DEVSS method]. We have four velocity gradient equations since this quantity is not symmetric like the stress tensor. The equation section of the GOMA input file is specified as follows:

```
Coordinate System = CARTESIAN
Element Mapping = isoparametric
```

```
    Mesh Motion = ARBITRARY
    Number of bulk species = 0
    Number of viscoelastic modes = 1

Number of EQ = 10
EQ = momentum1  Q2 U1  Q2     1.   1.   1.   1.   0.   0.
EQ = momentum2  Q2 U2  Q2     1.   1.   1.   1.   0.   0.
EQ = continuity Q1 P   Q1          1.             0.
EQ = stress11   Q1 S11 Q1     1.   1.   1.   1.   1.
EQ = stress12   Q1 S12 Q1     1.   1.   1.   1.   1.
EQ = stress22   Q1 S22 Q1     1.   1.   1.   1.   1.
EQ = gradient11 Q1 G11 Q1          1.             1.
EQ = gradient12 Q1 G12 Q1          1.             1.
EQ = gradient21 Q1 G21 Q1          1.             1.
EQ = gradient22 Q1 G22 Q1          1.             1.

                               ms   adv  bnd  dif  src   por
```

Here we are using biquadratic interpolation for the velocities (Q2) and bilinear interpolation (Q1) for everything else. The "`Number of viscoelastic modes`" card is adjusted from 1 for a single stress equation up to eight for a multimode stress equation. The input file in its entirety is given in Appendix 2.

Material File

Most of the important input parameters for viscoelastic flow can be found in the material file. The pertinent sections of the material file looks like:

```
---Physical Properties
Density                         = CONSTANT 0.0

---Mechanical Properties and Constitutive Equations
Solid Constitutive Equation     = NONLINEAR
Convective Lagrangian Velocity  = NONE
Lame MU                         = CONSTANT     1.
Lame LAMBDA                     = CONSTANT     1.
Stress Free Solvent Vol Frac    = CONSTANT     0.
Liquid Constitutive Equation    = NEWTONIAN
Viscosity                       = CONSTANT     0.0
Polymer Constitutive Equation   = OLDROYDB
Polymer Stress Formulation      = EVSS_F
Polymer Weight Function         = SUPG
Polymer Weighting               = CONSTANT      1.
Adaptive Viscosity Scaling      = CONSTANT      .0
Polymer Viscosity               = CONSTANT     1.
Polymer Time Constant           = CONSTANT     0.02
```

Since we are solving the problem in dimensionless form the density become the Reynold's number, Re, and it is set to zero for highly viscous fluids like polymers. The "`Liquid Constitutive Equation`" card is set to Newtonian for the solvent behavior. In this case, we have no solvent so the solvent viscosity is zero. We use the Oldroyd-B model for the "**Polymer Constitutive Equation**" since this can simplify to the Maxwell model if there is no solvent viscosity. We use the DEVSS formulation of Guenette and Fortin [1995], which is termed EVSS_F in the input deck. We use the streamline upwind Petrov Galerkin (SUPG) method developed by Brooks and Hughes [1982] to better handle the hyperbolic nature of the stress equation. The "**Polymer Weighting**" card is related to SUPG and should be set to the inverse of the average velocity. If it is set back to zero, the Galerkin formulation with no upwinding is recovered. The adaptive viscosity [Sun et al., 1999] is not being used for this example problem, so the "**Adaptive Viscosity Scaling**" is set to zero (This is an optional card and could have been omitted, but is just shown here for educational purposes.). The polymer viscosity is set to one, since the problem has been nondimensionalized. The "**Polymer Time Constant**" is actually the Weissenberg number and it will be the parameter we increase in our continuation strategy.

Continuation

The continuation strategy can be handled automatically in GOMA using zeroth or first order continuation. The section in the input deck for first order continuation would look like:

```
---
Continuation Specifications
---
Continuation                      = first
Continuation Type                 = MT
Boundary condition ID             = 24
Boundary condition data float tag = 1
Material id                        = 1
Material property tag              = 5000
Material property tag subindex     = 0
Initial parameter value            = 0.
Final parameter value              = 1.2
delta_s                            = .02
Maximum number of path steps       = 20
Minimum path step                  = .01
Maximum path step                  = 0.1
Continuation Printing Frequency    = 1
```

Details of how to set up a continuation problem can be found in the Advanced Capabilities GOMA Manual [Gates, et. al., in prep]. The section given here should allow you to do continuation on the Weissenberg number. The "**Continuation**" card can take four different key words: **zero** or **first** for single parameter continuation or **hzero** and **hfirst** for hunting continuation strategies when you are continuing in multiple variables. We are doing continuation on material property so we use MT for the "**Continuation Type**." The cards regarding boundary conditions are unused but required for the input parser. The "**Material id**" is the material number that the material property of interest comes from. The "**Material property tag**" is a code that points to the polymer time constant. The remaining cards control the step size choice and increment. Solutions at each Weissenberg number are placed in a single exodus file with the time stamp indicating the parameter value. The case that we have been looking at had a maximum Weissenberg number of about 0.65 and the zeroth order continuation seemed to work better at low Weissenberg numbers than the first order if we started our continuation from a Newtonian initial guess.

If you do not want to use the automatic continuation, you can accomplish zeroth order continuation by hand by using a lower parameter solution as an initial guess for a larger value of the parameter. Manual continuation is described in the beginner's tutorial GT-001.3.

Selected Results

Figure 3 shows a slice along the plane of contraction of the axial normal stress, τ_{xx}, as a function of Weissenberg number. From this plot, we can see that as the amount of elasticity is increased the peak stress at the reentrant corner increases (distance = 12) together with the stress in the downstream tube. This is to be expected since the exact solution for Poiseuille flow of a Maxwell fluid depends linearly on *We*. We can also see that the development length to achieve fully-developed flow increases with Weissenberg number and by *We* = 0.54 our mesh is too short to allow the stress to become fully-developed before the outflow is reached. This turns out to part of the cause of the loss of convergence at a *We* of 0.65 since the solution still looks well behaved. A more refined mesh in the downstream channel may also help the oscillations in the stress as the Weissenberg number is increased.

Figure 4 shows a slice along the plane of contraction of the normal stress, τ_{yy}, for two different Weissenberg numbers, one small (We = 0.05) and the other the maximum achievable on this mesh (We = 0.65). This component of the stress tensor is much better behaved than τ_{xx} since it changes little with the Weissenberg number. The only effect is to increase the stress peak at the reentrant corner by about 50% (from a value of 14 to 21).

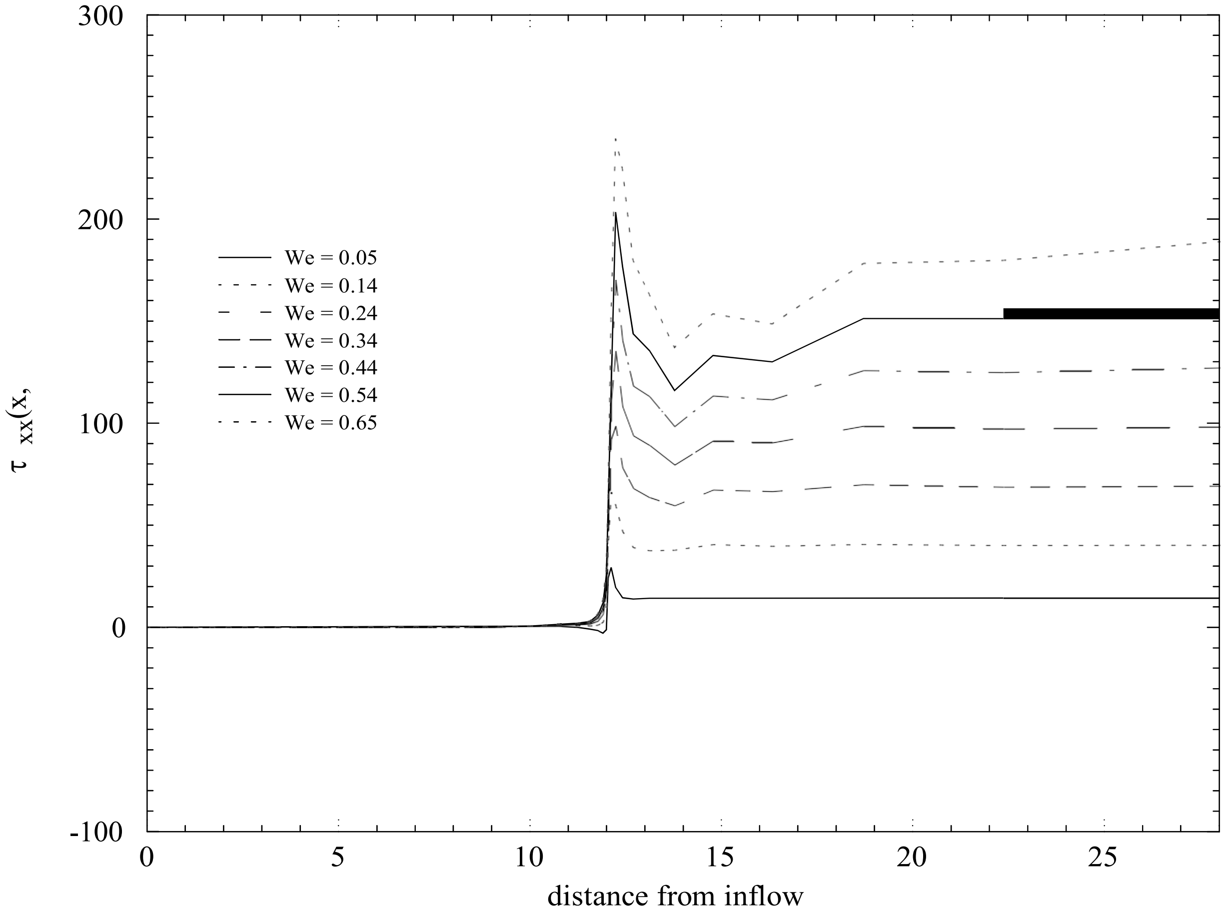


Figure 3. Normal stress, τ_{xx}, on the horizontal plane of contraction as a function of Weissenberg number.

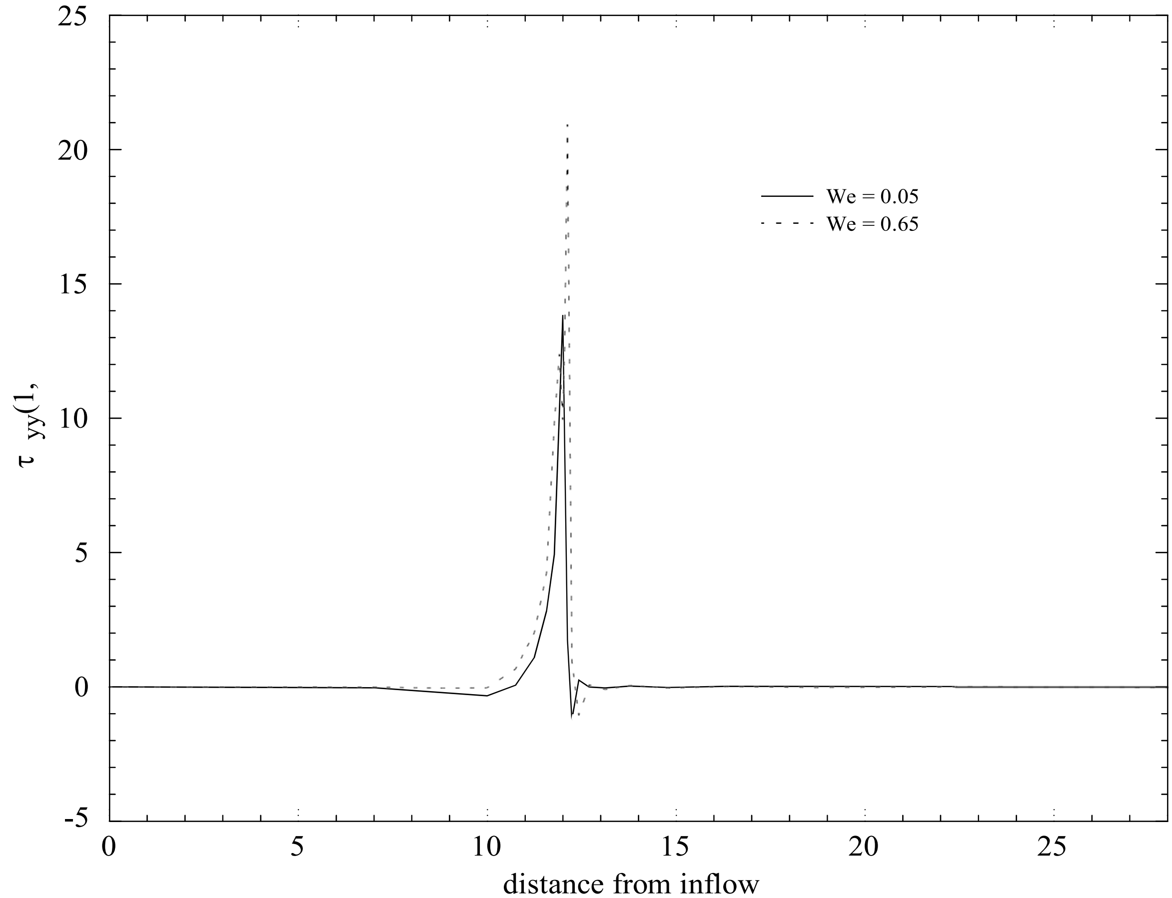
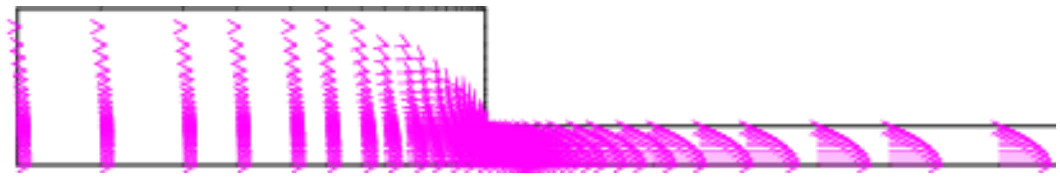


Figure 4. Normal stress, τ_{yy}, on the horizontal plane of contraction for a large and small Weissenberg number.

Figure 5 shows the velocity vectors for two different values of the Weissenberg number: a low and a high value. From this figure, we can see that the velocity field is relatively insensitive to the fluid elasticity, with the strongest effect being near the top of the large channel right before it enters the small channel. As the Weissenberg number is increased, we see the velocity field being pushed up into the large channel and obtaining a more curved appearance. This may be due to the vortex in the upper left hand corner that grows with Weissenberg number [Rao and Finlayson, 1992]. (It would be helpful to look at the streamlines as the fluid elasticity is increased, but at the time this memo was written the streamlines were not working for viscoelastic flow. This should be remedied soon.)

a)



b)

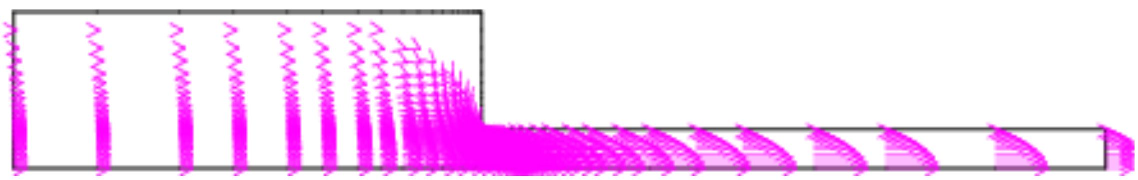


Figure 5. Velocity vectors for a small and large Weissenberg number: a) We = 0.05 b) We = 0.65

Extension to Discontinuous Galerkin Method for Stresses

To change the input deck for discontinuous Galerkin stresses from continuous stresses is quite simple. Just change the interpolation from bilinear continuous, Q1, to bilinear discontinuous, PQ1 for the stress variables. (Other discontinuous interpolations are available including P0 for piecewise constant, P1 is piecewise linear and PQ2 is piecewise quadratic.) An example using discontinuous Galerkin stresses is:

```
Number of EQ = 10
EQ = momentum1   Q2 U1   Q2      1.     1.    1.    1.    0.    0.
EQ = momentum2   Q2 U2   Q2      1.     1.    1.    1.    0.    0.
```

```
EQ = continuity Q1 P   Q1          1.             0.
EQ = stress11   PQ1 S11 PQ1    1.  1.   1.   1.   1.
EQ = stress12   PQ1 S12 PQ1    1.  1.   1.   1.   1.
EQ = stress22   PQ1 S22 PQ1    1.  1.   1.   1.   1.
EQ = gradient11 Q1 G11 Q1          1.             1.
EQ = gradient12 Q1 G12 Q1          1.             1.
EQ = gradient21 Q1 G21 Q1          1.             1.
EQ = gradient22 Q1 G22 Q1          1.             1.

                               ms  adv  bnd  dif  src  por
```

For the post processing specifications, you will also want to add a line that will interpolate the discontinuous stresses to the nodes for easy visualization.

```
Post Processing Specifications
Stress contours = yes
```

For the material file, you will also want to add the "`Discontinuous Jacobian Formulation`" card to ensure the off element Jacobian entries are included in the matrix. This greatly increases the radius of convergence for the method.

```
Polymer Weight Function= SUPG
Discontinuous Jacobian Formulation = FULL
Polymer Weighting= CONSTANT      1.
```

Extension to Multiple Modes

Adding more modes to the GOMA input is straightforward, but will be computationally expensive. For 4 modes or less, we recommend using the frontal solver and for 5 modes or more we recommend using either UMF or one of the iterative solvers. (Note, if an iterative solver is used, pressure stabilization must be used as well.) Change the "`Number of viscoelastic modes`" card to the desired number between zero and eight. Then make sure that you have input data for each mode: if you are using four modes, you will also need four viscosities on the viscosity card etc., *e.g.,*

```
Polymer Constitutive Equation  = GIESEKUS
Polymer Stress Formulation     = EVSS_F
Polymer Weight Function        = SUPG
Polymer Weighting              = CONSTANT     1.
Polymer Viscosity              = CONSTANT     1. 1. 1. 1.
Polymer Time Constant          = CONSTANT     0.02 0.02 0.02 0.02
Mobility Parameter             = CONSTANT     0.0 0.0 0.0 0.0
```

Extension to Axisymmetric Coordinates

When using axisymmetric coordinates, you will gain an extra stress variable and an extra velocity gradient variable. This is because there is a hoop stress and a component of the velocity gradient tensor in the θθ direction. This is what it would look like.

```
Coordinate System = CYLINDRICAL
```

```
    Element Mapping = isoparametric
    Mesh Motion = ARBITRARY
    Number of bulk species = 0
    Number of viscoelastic modes = 1

Number of EQ = 12
EQ = momentum1  Q2 U1  Q2     1.         1.   1.   1.   0.   0.
EQ = momentum2  Q2 U2  Q2     1.         1.   1.   1.   0.   0.
EQ = continuity Q1 P   Q1          1.                   0.
EQ = stress11   PQ1 S11 PQ1   1.         1.   1.   1.   1.
EQ = stress12   PQ1 S12 PQ1   1.         1.   1.   1.   1.
EQ = stress22   PQ1 S22 PQ1   1.         1.   1.   1.   1.
EQ = stress33   PQ1 S22 PQ1   1.         1.   1.   1.   1.
EQ = gradient11 Q1 G11 Q1                1.             1.
EQ = gradient12 Q1 G12 Q1                1.             1.
EQ = gradient21 Q1 G21 Q1                1.             1.
EQ = gradient22 Q1 G22 Q1                1.             1.
EQ = gradient33 Q1 G22 Q1                1.             1.

                               ms   div  adv  bnd  dif  src  por
```

Adaptive Viscosity

To use the adaptive viscosity [Sun, et al., 1999], you can just put a nonzero value on the adaptive viscosity card. In Sun's paper, this parameter is referred to as δ, which is between zero and one [see page 289, Sun, et al., 1999]:

$$0 \le \delta \le 1$$

In the input deck we have arbitrarily chosen 0.5 as an appropriate value since it fits the above criterion.

```
Polymer Constitutive Equation  = OLDROYDB
Polymer Stress Formulation     = EVSS_F
Polymer Weight Function        = SUPG
Polymer Weighting              = CONSTANT     1.
Adaptive Viscosity Scaling     = CONSTANT     .5
Polymer Viscosity              = CONSTANT    1.
Polymer Time Constant          = CONSTANT    0.02
```

Conclusions

We hope that you get a lot of use out of the viscoelastic models in GOMA and that this tutorial memo is helpful in explaining a complicated, and mathematically intensive subject. Please feel free to email or call if you have questions or comments or would like further clarification on one of the numerical methods. Reading the papers listed in the references may also be helpful.

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Appendix 1: FASTQ File for Mesh

```
TITLE First VE test problem

POINT     1      0.      0.
POINT     2     16.      0.
POINT     3     16.      1.
POINT     4      0.      1.
POINT     5      0.      4.
POINT     6    -12.      4.
POINT     7    -12.      1.
POINT     8    -12.      0.
LINE      1   STR     1     2     0   12  1.538
LINE      2   STR     2     3     0    6  .8
LINE      3   STR     3     4     0   12  .65
LINE      4   STR     4     1     0    6  1.25
LINE      5   STR     4     5     0    8  1.4
LINE      6   STR     5     6     0   10  1.538
LINE      7   STR     6     7     0    8  .71429
LINE      8   STR     7     8     0    6  1.25
LINE      9   STR     8     1     0   10  .65
LINE     10   STR     7     4     0   10  .65
REGION    1      1    -1  -2 -3 -4
REGION    2      1    -4  -10 -8 -9
REGION    3      1    -10 -5 -6 -7
SCHEME    0 m
BODY       1 2 3
NODEBC    1      1  9
NODEBC    2      2
NODEBC    3      3  5  6
NODEBC    4      7  8
ELEMBC   10      1  9
ELEMBC   20      2
ELEMBC   30      3  5  6
ELEMBC   40      7  8
POINBC  100      3
NINE
EXIT
```

Appendix 2: Sample Viscoelastic Input Deck

```
-------------------------------------------------------------
                 FEM Problem Specifications
-------------------------------------------------------------
FEM file                         = newt.exoII
Output EXODUS II file            = out.exoII
GUESS file                       = contin.dat
SOLN file                        = soln.dat
Write intermediate results       = no
-------------------------------------------------------------
                 General Specifications
-------------------------------------------------------------
Number of processors             = 1
Output Level                     = 0
Debug                            = 0
Initial Guess                    = read_exoII
-------------------------------------------------------------
                 Time Integration Specifications
-------------------------------------------------------------
Time integration                 = steady
delta_t                          = 5.0e-6
Maximum number of time steps     = 250
Maximum time                     = 10.0e+0
Minimum time step                = 1.0e-12
Time step parameter              = 0.
#Time step error                 = err=.008  mesh=1  v=1  T=0   C=0  p=0
Time step error                  = .008  1   1   0   0   0
Printing Frequency               = 1
-------------------------------------------------------------
                 Solver Specifications
-------------------------------------------------------------
Solution Algorithm               = umf
UMF XDIM                         = 10330088
UMF IDIM                         = 10330088
Preconditioner                   = ls
Polynomial                       = 1
Size of Krylov subspace          = 64
Orthogonalization                = classical
Maximum Linear Solve Iterations  = 1000
Number of Newton Iterations      = 12
Newton correction factor         = 1.
Normalized Residual Tolerance    = 1.0e-10
Residual Ratio Tolerance         = 1.0e-2
-------------------------------------------------------------
   Boundary Condition Specifications
-------------------------------------------------------------
Number of BC = -1
BC = V     NS     1       0.
BC = V     NS     2       0.
BC = U     NS     3       0.
BC = V     NS     3       0.
BC = V     NS     4       0.
```

```
BC = GD_LINEAR SS 40 R_MOMENTUM1  0 VELOCITY1      0  0.   1.
BC = GD_PARAB  SS 40 R_MOMENTUM1  0 MESH_POSITION2 0 -1.5  0. 0.09375
BC = GD_LINEAR SS 20 R_MOMENTUM1  0 VELOCITY1      0  0.   1.
BC = GD_PARAB  SS 20 R_MOMENTUM1  0 MESH_POSITION2 0 -6    0. 6.
BC = S22     NS     4     0.
BC = P     NS    100    0.
BC = GD_LINEAR SS 40 R_STRESS11  0 POLYMER_STRESS11 0  0. 1.
BC = GD_PARAB SS 40 R_STRESS11  0 MESH_POSITION2 0 0.  0. -0.00140625
END OF BC


----
Problem Description
---

Number of Materials = 1

MAT = ve   1

   Coordinate System = CARTESIAN

   Element Mapping = isoparametric

   Mesh Motion = ARBITRARY

   Number of bulk species = 0
   Number of viscoelastic modes = 1


Number of EQ = 10
EQ = momentum1  Q2U1Q2     1.    1.   1.   1.   0.   0.
EQ = momentum2  Q2U2Q2     1.    1.   1.   1.   0.   0.
EQ = continuityQ1P Q1           1.             0.
EQ = stress11   Q1S11Q1     1.    1.   1.   1.   1.
EQ = stress12   Q1S12Q1     1.    1.   1.   1.   1.
EQ = stress22   Q1S22Q1     1.    1.   1.   1.   1.
EQ = gradient11 Q1G11Q1           1.             1.
EQ = gradient12 Q1G12Q1           1.             1.
EQ = gradient21 Q1G21Q1           1.             1.
EQ = gradient22 Q1G22Q1           1.             1.

                                               ms   adv  bnd  dif  src  porous

Post Processing Specifications

Stream Function = yes
Streamwise normal stress = no
Pressure contours = yes
Second Invarient of Strain = no
Mesh Dilatation = no
Navier Stokes Residuals = no
Moving Mesh Residuals = no
```

```
Mass Diffusion Vectors = no
Mass Fluxlines = no
Energy Conduction Vectors = no
Energy Fluxlines = no
Time Derivatives = no
Mesh Stress Tensor = no
Mesh Strain Tensor = no
```

Appendix 3: Sample Viscoelastic Material File

```
Material Data File for SAMPLE

/**********Form of each property card *********/
PROPERTY   = MODEL   FLOAT#1 FLOAT#2 .....FLOAT#5

/**********Form of each Constitutive card *********/
MECHANICS_TYPE   = MODEL

Of the many available options, MODEL can be USER
/************************************************/


---Physical Properties
Density                          = CONSTANT 0.0

---Mechanical Properties and Constitutive Equations
Solid Constitutive Equation      = NONLINEAR
Convective Lagrangian Velocity   = NONE
Lame MU                          = CONSTANT 1.
Lame LAMBDA                      = CONSTANT      1.
Stress Free Solvent Vol Frac     = CONSTANT   0.
Liquid Constitutive Equation     = NEWTONIAN
Viscosity                        = CONSTANT 0.0
Polymer Constitutive Equation    = GIESEKUS
Polymer Stress Formulation       = EVSS_F
Polymer Weight Function          = SUPG
Polymer Weighting                = CONSTANT      .1
Adaptive Viscosity Scaling       = CONSTANT      .0
Polymer Viscosity                = CONSTANT   1.
Polymer Time Constant            = CONSTANT   0.02
Mobility Parameter               = CONSTANT      0.0
Surface Tension                  = CONSTANT      60.

---Thermal Properties
Conductivity                       = CONSTANT  1.
Heat Capacity                      = CONSTANT  1.
Volume Expansion                   = CONSTANT 1.
Reference Temperature            = CONSTANT   0.
Liquidus Temperature             = CONSTANT   1.
Solidus Temperature              = CONSTANT   1.
```

```
---Electrical Properties
Electrical Conductivity          = CONSTANT      1

---Microstructure Properties
Media Type                       = CONTINUOUS
Permeability                     = CONSTANT1.


---Species Properties
Diffusion Constitutive Equation = FICKIAN
Diffusivity                      =   CONSTANT   0   1.
Latent  Heat  Vaporization     = CONSTANT   0        0.
Latent Heat Fusion               = CONSTANT  0       0.
Vapor Pressure                   = CONSTANT  0       0.
Reference Concentration          = CONSTANT  0       0.
                                                     ^
********Species Number**********************|

----Source Terms
Navier-Stokes Source             = CONSTANT    0.  0.  0.
Solid Body Source                = CONSTANT    0.  0.  0.
Mass Source                      = CONSTANT    0.
Heat Source                      = CONSTANT    0.
Species Source                   = CONSTANT    0.  2.
Current Source                   = CONSTANT    0.
```